## THE UNIVERSITY OF AKRON

## Mathematics and Computer Science

Lesson 4: Basic Algebra, Part II

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$$
\begin{gathered}
\mathbb{N}^{4} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \\
a^{3} a^{4}=a^{7}(a b)^{10}=a^{10} b^{10} \\
-(a b-(3 a b-4))=2 a b-4 \\
(a b)^{3}\left(a^{-1}+b^{-1}\right)=(a b)^{2}(a+b) \\
(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} \\
2 x^{2}-3 x-2=(2 x+1)(x-2) \\
\frac{1}{2} x+13=0 \Longrightarrow x=-26 \\
\mathrm{G}=\{(x, y) \mid y=f(x)\} \\
\mathbb{S}
\end{gathered}
$$

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## 4. Basic Algebra, Part II

### 4.1. The Distributive Law

In the process of manipulating algebraic expressions, we often find ourselves multiplying expression wherein one or both factors are sums. The Distributive Law tells us how to properly handle products involving sums.

The Distributive Law:
Let $a, b$, and $c$ represent numbers. Then

$$
\begin{equation*}
a(b+c)=a b+a c \tag{1}
\end{equation*}
$$

In the age of the computer, we might say that the Distributive Law is a statement of how multiplication interfaces with addition. It is one of the major tools used in algebra.

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Using the formula

$$
\begin{equation*}
a(b+c)=a b+a c \tag{2}
\end{equation*}
$$

from left-to-right might be referred to as expanding or distributing through and using it from right-to-left is referred to as factoring.

Reading formula (2) from left-to-right, we can see that when multiplying one factor, $a$, by the sum of two quantities, $b$ and $c$, we simply multiply $a$ by each term of the sum, and add the result. This is referred to as distributing $a$ through the sum.

Equation (2) is a property of our arithmetic system. Is it true that

$$
4(2+5)=4 \cdot 2+4 \cdot 5 ?
$$

Of course, both sides evaluate to 28 . This property of our arithmetic system-rarely used when dealing with only numerical values-is extremely useful with dealing with symbolic quantities.

Time Out! In the previous paragraph I made the statement that the distributive law is "rarely used when dealing with only numerical values." Is that a true statement? E-mail me if you have an example of the distributive law in everyday life! D. P. Story.

Illustration 1. Expand each using (1).
(a) $x(y+z)=x y+x z$ and $x(y-z)=x y-x z$. Of course we can expand or distribute through even when there is subtraction present.
(b) $x^{3}\left(x^{5}+4 x^{3}+2\right)=x^{8}+4 x^{6}+2 x^{3}$, where we have distributed through $x^{3}$ and combined the powers of $x$ using the first LAW of Exponents.
(c) $w t\left(w^{3} t-5 w^{-1} t^{2}\right)=w^{4} t^{2}-5 t^{3}$. Once again we combine similar powers by adding exponents per the first Law of Exponents.
(d) The $a, b$, and $c$ in (2) may be complex algebraic expressions themselves:

$$
(x+y)(a+b)=(x+y) a+(x+y) b .
$$

Here, we have distributed the factor $(x+y)$ through the sum; alternately, we can distribute the $(a+b)$ through the sum:

$$
(x+y)(a+b)=(a+b)(x+y)=(a+b) x+(a+b) y
$$

These are just simple applications of the Distributive Law.
(e) Continuation. Usually, when we have an expression of the form $(x+y)(a+b)$, the task is to "multiply it out." Let's continue that process-all we do is to apply the Distributive Law again:

$$
\begin{array}{rlr}
(x+y)(a+b) & =(x+y) a+(x+y) b & \triangleleft \text { Dist. Law } \\
& =(x a+y a)+(x b+y b) & \triangleleft \text { Dist. Law } \\
& =x a+y a+x b+y b &
\end{array}
$$

The point is that all you do is carefully apply the Distributive LAW when ever you are multiplying out algebraic expressions.
Illustration Notes: Expanding expressions of the type discussed in examples (d) and (e) above will be taken up in Lesson 5.

Exercise 4.1. Expand each of the following expressions completely, combining similar exponentials using the Law of Exponents wherever appropriate. Slowly and methodically work out your solutions first using good notation and techniques.
(a) $a b(c+d)$
(b) $w^{-1}\left(4 w-3 w^{2}\right)$
(c) $x^{3}\left(\sqrt{x}+x^{3 / 2}\right)$
(d) $\sqrt{w}(\sqrt{w}+3)$ (e) $s^{3}\left(5 s^{-4}-\frac{3}{s^{2}}\right)$
(f) $(s t)^{2}\left(s^{-1}+t^{-1}\right)$

Exercise 4.2. Simplify

$$
s^{-1 / 2} t^{1 / 2}\left[\frac{1}{(s t)^{1 / 2}}+(s t)^{1 / 2}\right]
$$

Using the Distributive Law and the Law of Exponents, eliminating all negative exponents.

The formula $a(b+c)=a b+a c$ when read from right-to-left is used in two ways: combining similar terms and simple factoring

We can reason as follows. The expression $a b+a c$ has two terms. Notice that each term has a factor of $a$. (Here we say that $a$ is a common
factor to all the terms.) We can factor out this common factor to obtain,

$$
a b+a c=a(b+c)
$$

In the illustration below, we use this formula in the following form:

$$
\begin{equation*}
b a+c a=(b+c) a . \tag{3}
\end{equation*}
$$

Same formula, I've just written the common factor on the right, and factored it out to the right.

Illustration 2. Combine similar terms using (3).
(a) $4 x+6 x=(4+6) x=10 x$. The first equality comes from the Distributive Law, read from right-to-left; in particular equation (3).
(b) $5 w^{3}-2 w^{3}=(5-2) w^{3}=3 w^{3}$, by equation (3).
(c) $4 x^{3 / 2}+6 x^{3 / 2}-8 x^{3 / 2}=2 x^{3 / 2}$.
(d) After reviewing the two previous examples, you should realize that there is nothing difficult about the operation of adding similar terms together. The process of combining can be accelerated: $4 s t+12 s t=16 s t$.

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$$
\begin{aligned}
& \text { (e) } \sqrt{x+1}-\frac{1}{2} \sqrt{x+1}=\frac{1}{2} \sqrt{x+1} \text {. } \\
& \text { (f) } \frac{5}{w^{2}+1}+\frac{4}{w^{2}+1}=\frac{9}{w^{2}+1} \text {. }
\end{aligned}
$$

Exercise 4.3. Combine similar terms of each of the following.
(a) $4 w-12 w$
(b) $23 p^{2} q^{3}-6 p^{2} q^{3}$
(c) $\frac{1}{2} t^{3}+\frac{3}{2} t^{3}$
(d) $\frac{3}{s^{2}}-\frac{6}{s^{2}}$
(e) $\frac{4 w}{\sqrt{s}}-\frac{3 w}{\sqrt{s}}$
(f) $4 x y-5 s t$

The Distributive Law can be used in a more general way than was just illustrated. Reading the Distributive Law from right-to-left can be thought of as factoring out all factors common to all terms of an expression.

Illustration 3. Using (3), factor out any factors common to all the terms. The Law of Exponents is used throughout these examples.
(a) $x y^{3}-x y^{2}=x y^{2}(y-1)$. The validity of this step can be verified by distributing $x y^{2}$ back through $y-1$.
(b) $4 s^{2} t^{3}-8 s t=4 s t\left(s t^{2}-2\right)$. Again, multiply back through to verify.

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(c) $2 x^{3 / 2}+4 x^{5 / 2}=2 x^{3 / 2}(1+2 x)$. Here, we have factored $x^{5 / 2}$ as $x^{5 / 2}=x^{3 / 2} x$, by the Law of Exponents, then factored out a common factor of $x^{3 / 2}$.
(d) Here's an example with more detail thrown in.

$$
\begin{array}{rlrl}
4 x^{3} y^{6} z-12 x^{4} y^{7} & =\left(4 x^{3} y^{6}\right) z-\left(4 x^{3} y^{6}\right)(3 x y) & & \triangleleft \text { Law Exp. } \# 1 \\
& =\left(4 x^{3} y^{6}\right)[z-(3 x y)] & & \triangleleft \text { Distrib. Law } \\
& =4 x^{3} y^{6}(z-3 x y) & & \\
\text { (e) } \frac{s t^{3}}{w^{2}}-\frac{s t}{w^{2}}=\frac{s t}{w^{2}}\left(t^{2}-1\right)=\frac{s t\left(t^{2}-1\right)}{w^{2}} . & & \\
\text { (f) } \frac{x}{w}-\frac{y}{w}=\frac{1}{w}(x-y) & =\frac{x-y}{w}
\end{array}
$$

Illustration Notes: Examples (e) and (f) are tantamount to adding together two fractions having the same denominator.
Operational Tips. Here are two observations that may aid you in factoring out using the Distributive Law.

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$\triangleright$ Tip 1: When factoring out common factors involving exponents, factor out the powers having the smallest exponent:

$$
x^{4} y^{3}+x^{2} y^{5}=x^{2} y^{3}\left(x^{2}+y^{2}\right)
$$

Here, I factored out $x^{2}$ because it was the smallest power of $x$ and I factored out $y^{3}$ because it was the smallest power of $y$.
$\triangleright$ Tip 2: Notice that when factoring out the lowest common powers

$$
x^{8} y^{5}+x^{3} y^{7}=x^{3} y^{5}\left(x^{8-3} y^{5-5}+x^{3-3} y^{7-5}\right)=x^{3} y^{5}\left(x^{5}+y^{2}\right)
$$

the calculation of the powers of the remaining factors is made by subtracting off the factored power, as illustrated above.

Review Illustration 3 and observe these Operational Tips in action. Then apply them to the following ...

Exercise 4.4. Factor out all factors that are common to all the terms of the algebraic expressions below.
(a) $3 w^{3} v^{4}+9 w^{2} v^{2}$
(b) $x^{2} \sqrt{y}+x^{3} y^{3 / 2}$
(c) $a b^{2} c^{3}+a^{3} b^{2} c$
(d) $\frac{x y}{5}-\frac{x y}{10}$
(e) $(x+1) \sqrt{x}-x \sqrt{x}$
(f) $\frac{2 p^{8} q^{4}}{w^{2}}+\frac{4 p^{8}}{w}$

The Operational Tips are still valid for negative powers as well. Dealing with negative powers is when students find themselves most at risk for errors. The rules are simple: Factor out the smallest power (Tip 1), by subtracting it form the others (Tip 2).

Note: It is subtracting a negative number that causes the problems: Remember,

$$
\begin{equation*}
2-\left(-\frac{1}{2}\right)=2+\frac{1}{2}=\frac{5}{2} \tag{4}
\end{equation*}
$$

a "minus times a minus is a plus."
Illustration 4. Factor out any common factors present.
(a) $4 x^{2}-2 x^{-1 / 2}=2 x^{-1 / 2}\left(2 x^{5 / 2}-1\right)$. The details of the calculation of the exponent is given in equation (4).
(b) $3 x^{1 / 2} y^{-2 / 3}-12 x^{-3 / 2} y^{-5 / 3}=3 x^{-3 / 2} y^{-5 / 3}\left(x^{2} y-4\right)$. Verify that the exponents in $x^{2}$ and $y$ are the correct ones as calculated by Tip 2.

Let's have one more set of exercises to illustrate this last point.

Exercise 4.5. While taking the advice of the Operational Tips, factor out all common factors.
(a) $x^{8} y^{-2}-x^{-3} y^{-4}$
(b) $6 a^{3 / 2}-9 a^{-1 / 2}$
(c) $12 w^{-3 / 2} s^{-12 / 5}+8 w^{-1 / 2} s^{-2 / 5}$

### 4.2. On the Cancellation of Factors

What do I mean by that? I refer to the rather profound result

$$
\frac{a}{a}=1 \quad a \neq 0
$$

Most often this fact is used in the following context:

$$
\frac{a b}{a c}=\frac{b}{c}
$$

and students, operationally, give it this action,

$$
\frac{a b}{a c}=\frac{\not d b}{\phi c}=\frac{b}{c},
$$

though I would never do that myself.

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Let's elevate this to the status of a shadow box.

$$
\begin{align*}
& \text { CANCELLATION LAW } \\
& \qquad \frac{a b}{a c}=\frac{b}{c} \quad a \neq 0, c \neq 0 \tag{5}
\end{align*}
$$

The so-called Cancellation Law has already been visited by these lessons. When we discussed the Law of Exponents in Lesson 2, we advertised a version of Law $\# 1$ as a Cancellation Law. (How's that for fancy hypertexting?)

We have already seen that

$$
\frac{a^{n}}{a^{m}}=a^{n-m}
$$

Our new Cancellation Law is just the case of $n=m=1$. Consequently, very little time will be devoted to cancellation. What we will do is to concentrate on cancellation after factoring.

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The Cancellation Law to can be verbalized as follows:

$$
\frac{a b}{a c}=\frac{b}{c}
$$

states that if $a$ is a factor of the numerator and the denominator, then the $a$ can be cancelled from the expression.

Here are a few examples.
Illustration 5. Simplify each of the following.
(a) $\frac{x y-x y^{2}}{x y}=\frac{(x y)(1-y)}{x y}=1-y$, where we have cancelled the factor $x y$ from the numerator and denominator.
(b) $\frac{\sqrt{x}+x \sqrt{x}}{x \sqrt{x}}=\frac{\sqrt{x}(1+x)}{x \sqrt{x}}=\frac{1+x}{x}$. Here's an important point.

Some students will cancel the $x$ 's as well. This is illegal. The reason for this is that $x$ is not a common factor of both the numerator and denominator.
(c) $\frac{x^{2} y^{3} z^{5}-4 x^{3} y^{2} z^{5}}{x y^{4} z^{5}}=\frac{x^{2} y^{2} z^{5}(y-4 x)}{x y^{4} z^{5}}=\frac{x(y-4 x)}{y^{2}}$.

Illustration Notes: You must train your eyes to see common factors, automatically factor them out, and cancel if appropriate. This is a very important skill.
Exercise 4.6. Simplify each of the following fractions by factoring out common factors and cancelling.
(a) $\frac{a+a b}{a}$
(b) $\frac{x^{2} y^{4} z^{9}}{x^{3} y^{5} z^{3}}$
(c) $\frac{x^{-3} y^{-2}}{x^{4} y^{-5}}$
(d) $\frac{3 x^{3}+9 x^{2}}{3 x^{3}}$
(e) $\frac{4 x^{5 / 2}-x^{7 / 2}}{x^{2}}$
(f) $\frac{y^{3} \sqrt{x}-y^{2} \sqrt{x}}{x y^{2} \sqrt{x}}$

Don't forget, negative powers can be factored out as well.
Exercise 4.7. Simplify each of the following fractions by factoring out common factors and cancelling. After factoring and cancelling, remove all negative exponents.

$$
\text { (a) } \frac{x^{-3} y^{-2}-x^{2} y^{-5}}{x^{4} y^{-5}} \text { (b) } \frac{4 x^{-5 / 2}-x^{7 / 2}}{x^{1 / 2}} \text { (c) } \frac{x^{2}}{4 x^{3}+2 x^{-3}}
$$

### 4.3. How to Multiply Fractions

Multiplying fractions together est du gâteau. ${ }^{1}$ We have been multiplying fractions throughout these lessons. For any $a, b, c$ and $d$, we have

$$
\begin{equation*}
\frac{a}{b} \frac{c}{d}=\frac{a c}{b d} \tag{6}
\end{equation*}
$$

When multiplying two fractions together, just multiply the numerators and divide by the product of the denominators.

## Illustration 6.

(a) $\frac{1}{2} \frac{x}{x+y}=\frac{x}{2(x+y)}$.
(b) $\frac{x^{3} y}{(x+y)} \frac{z^{2}}{(x+y)}=\frac{x^{3} y z^{2}}{(x+y)^{2}}$.

[^0](c) Any number or symbol can be thought of as a fraction; for example, $4=\frac{4}{1}$ and $x=\frac{x}{1}$. In light of this observation, consider the following example:
$$
4 \cdot \frac{x}{2 x-1}=\frac{4}{1} \cdot \frac{x}{2 x-1}=\frac{4 x}{2 x-1} .
$$

This simplification is usually accelerated; we jump from the first to the last expression in one step.
(d) The generalization of the previous point would be, for any $a$, and $b \neq 0$,

$$
\begin{equation*}
\frac{a}{b}=a \cdot \frac{1}{b} \tag{7}
\end{equation*}
$$

(e) $(x+1) \cdot \frac{y}{z} \cdot \frac{x}{(x+1)}=\frac{(x+1) y x}{z(x+1)}=\frac{x y}{z}$, where we have cancelled the common factor $(x+1)$.

Time for you to chime in. Try the following ...

Exercise 4.8. Combine all fractions, simplify as appropriate.
(a) $\frac{4 x^{2}}{5 y^{3}} \frac{x^{4} y^{2}}{x^{2} y}$
(b) $(3 x-2)^{2} \frac{4 x}{(x+1)^{3}}$
(c) $\frac{x}{y}\left[\frac{y}{z}+\frac{1}{2}\right]$
(d) $\left[\frac{x^{4}}{(x+3)} \cdot \frac{x}{2}\right]^{-1}$
(e) $\left[x\left(\frac{x}{y}\right)^{-1}\right]^{-3}$

Before leaving this sections, let's discuss ratios of fractions; that is, expressions of the form

$$
\frac{a / b}{c / d}=\frac{\frac{a}{b}}{\frac{c}{d}}
$$

The question is how can this be simplified? This question is easily answered using the techniques of this lesson. Observe that

$$
\frac{\frac{a}{b}}{\frac{b}{c}}=\underbrace{\frac{a}{b} \cdot \frac{1}{c}}_{\text {from (7) }}=\underbrace{\frac{a}{b} \cdot \frac{d}{c}}_{\text {from (8) }}=\underbrace{\frac{a d}{b c}}_{\text {by }(6)}
$$

Thus, the governing equation for simplifying multifractions is

$$
\begin{equation*}
\frac{a / b}{c / d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a d}{b c} \tag{8}
\end{equation*}
$$

ExERCISE 4.9. Simplify each of the following using (8).

$$
\text { (a) } \frac{x /(x+y)}{x^{2} /(x+y)^{2}} \text { (b) } \frac{x^{3} y^{2} / s^{9} t^{2}}{x^{4} y^{8} / s^{10}} \text { (c) } \frac{x^{4}}{x^{3} / 4} \text { (d) } \frac{x y /(x+y)}{(x+y)}
$$

- Rationalizing the Denominator

The traditional rules for simplifying algebraic expressions frown on having radicals in the denominator. There is a simple device for appeasing the rules in this case it's called rationalizing the denominator: For example,

$$
\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}
$$

That is, we multiply the numerator and denominator by the appropriately chosen radical for the purpose of ridding ourselves of radicals in the denominator.

Illustration 7. Rationalize the denominator in each of the following.
(a) $\frac{4}{\sqrt{3}}=\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}=\frac{4 \sqrt{3}}{3}$.
(b) Rationalizing is not restricted to numerical expressions.

$$
\frac{21 y}{\sqrt{x}}=\frac{21 y}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}=\frac{21 y \sqrt{x}}{x} .
$$

(c) $\frac{4 x y}{\sqrt{x^{2}+1}}=\frac{4 x y}{\sqrt{x^{2}+1}} \cdot \frac{\sqrt{x^{2}+1}}{\sqrt{x^{2}+1}}=\frac{4 x y \sqrt{x^{2}+1}}{x^{2}+1}$.
(d) This trick is not restricted to the square root either:

$$
\frac{5}{\sqrt[3]{2}}=\frac{5}{2^{1 / 3}} \cdot \frac{2^{2 / 3}}{2^{2 / 3}}=\frac{5\left(2^{2 / 3}\right)}{2}=\frac{5 \sqrt[3]{4}}{2}
$$

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ExERCISE 4.10. Rationalize the denominator of each of the following.
(a) $\frac{x}{\sqrt{7}}$
(b) $\frac{3 x^{2}}{\sqrt{x+1}}$
(c) $\frac{x y}{\sqrt[4]{3}}$

### 4.4. How to Add Fractions

Adding fractions is always a sore point among students, but it really isn't that difficult. Here is the guiding principle: When adding two fractions with the same denominator, simply add the numerators and divide by the common denominator. In symbols:

Addition of Fractions:

$$
\begin{equation*}
\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \tag{9}
\end{equation*}
$$

This is the simplest situation. Let's look at some examples.
Illustration 8. In each case, we already have a common denominator; consequently, adding or subtracting fractions is trivial pursuit.

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(a) $\frac{3}{9}+\frac{5}{9}=\frac{8}{9}$. The two fractions have a common denominator, so we add the numerators. This is a skill we have been working all our lives to perfect.

$$
\begin{aligned}
& \text { (b) } \frac{3 x+4}{5}+\frac{6 x-5}{5}=\frac{(3 x+4)+(6 x-5)}{5}=\frac{9 x-1}{5} \\
& \text { (c) } \frac{4 s}{s+t}-\frac{5 s}{s+t}=\frac{4 s-5 s}{s+t}=\frac{-s}{s+t}=-\frac{s}{s+t}
\end{aligned}
$$

The next question is, how do you add fractions when don't have a common denominator? The answer is that you create a common denominator using a simple device. Next up is an enumeration of the steps to add to fractions together having different denominators.

Problem. Add $\frac{x}{s}+\frac{y}{t}$. (Actually, combine would be a better termthey are already added!)
$\triangleright$ Step 1: Multiply the first fraction by $t / t$ and the second fraction by $s / s$ to obtain

$$
\frac{x}{s}+\frac{y}{t}=\frac{x}{s} \frac{t}{t}+\frac{y}{t} \frac{s}{s} .
$$

$\triangleright$ Step 2: Combine fractions for each term.

$$
\frac{x}{s}+\frac{y}{t}=\frac{x t}{s t}+\frac{y s}{t s}
$$

They now have a common denominator.
$\triangleright$ Step 3: Add the fractions together.

$$
\frac{x}{s}+\frac{y}{t}=\frac{x t+y s}{s t}
$$

The idea is to multiply the numerator and denominator by the same quantity in such a way that, at the end of the process, all fractions have the same denominator.

Example 4.1. Combine each of the following. The common denominator is obtained by taking the product of the denominators of each term.

$$
\text { (a) } 6 x-\frac{3}{x} \text { (b) } \frac{1}{\sqrt{x}}-\sqrt{x} \text { (c) } \frac{x^{2}}{x+y}-\frac{x}{y} \text { (d) } \frac{2}{x}-\frac{3 y}{a}+\frac{z}{2 b}
$$

Here are a few exercises for you to practice on.
Exercise 4.11. Combine each expression.

$$
\text { (a) } \frac{2}{a}+\frac{a}{2} \text { (b) } \frac{3 x}{2 x-y}+\frac{y}{x} \text { (c) } \frac{1}{2 y}-\frac{x}{3}+\frac{y}{(1+x)}
$$

- How to get a Least Common Denominator

When combining the sum of algebraic expressions it is not always wise or efficient to obtain a common denominator by simply multiplying the denominators of each term together. (See the techniques illustrated in the previous paragraph on How to Add Fractions.) Most of the time we want to obtain the least common denominator; that is, a denominator that is a simple as possible.

Illustration 9. Contrast the following two methods of combining the expression $\frac{1}{x}+\frac{2}{x^{2}}+\frac{3}{x^{3}}$.
(a) Get a common denominator by multiplying the denominators together: $x \cdot x^{2} \cdot x^{3}=x^{6}$. Then

$$
\begin{align*}
\frac{1}{x}+\frac{2}{x^{2}}+\frac{3}{x^{3}} & =\frac{1}{x} \frac{x^{5}}{x^{5}}+\frac{2}{x^{2}} \frac{x^{4}}{x^{4}}+\frac{3}{x^{3}} \frac{x^{3}}{x^{3}} \\
& =\frac{x^{5}}{x^{6}}+\frac{2 x^{4}}{x^{6}}+\frac{3 x^{3}}{x^{6}}=\frac{x^{5}+2 x^{4}+3 x^{3}}{x^{6}} \tag{10}
\end{align*}
$$

(b) An alternate choice of a common denominator is $x^{3}$. In this case we get

$$
\begin{align*}
\frac{1}{x}+\frac{2}{x^{2}}+\frac{3}{x^{3}} & =\frac{1}{x} \frac{x^{2}}{x^{2}}+\frac{2}{x^{2}} \frac{x}{x}+\frac{3}{x^{3}} \\
& =\frac{x^{2}}{x^{2}}+\frac{2 x}{x^{3}}+\frac{3}{x^{3}}=\frac{x^{2}+2 x+3}{x^{3}} \tag{11}
\end{align*}
$$

Illustration Notes: Both equations (10) and (11) are valid, but give dramatically different results. Obviously, (10) is unduly complex; in fact, $x^{3}$ is a common factor to both numerator and denominator in (10) and should be cancelled.

- The reason for the disparity between these two answers is the choice is the denominator. It was a bad choice to use $x^{6}$. The choice of $x^{3}$ is referred to as the least common denominator.

Finding the Least Common Denominator

1. Factor each denominator completely.
2. Write down a list of all factors.
3. Among all factors having the same base, remove all but the one having the highest exponent.
4. Multiply all the factors together left after step 3 to obtain the least common denominator.

Next is an important example that explains in detail how to use this LCD algorithm. You should read and study this example.

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Example 4.2. Combine $\frac{1}{4 x}-\frac{4}{x^{3} y}+\frac{3 x}{2 y^{4}}$.
If you have a history of not being able to successfully compute the LCD of an algebraic expression and carry out the process of adding fractions together, go slowly through out the rest of this section. Be sure you understand the process.

The LCD Algorithm works equally well with radicals and fractional exponents.

Example 4.3. Combine the expression: $\frac{2 \sqrt{x}}{5 y^{2}}+\frac{3}{2 x^{1 / 2} y^{3 / 2}}$.
Let's test your understanding of the algorithm for obtaining the least common denominator. Below is a little quiz for you to take. Consider the questions carefully before responding. Follow the steps of the LCD Algorithm.

Quiz. Determine the LCD for each of the following.

1. $\frac{3 x}{2 y^{2} z^{3}}-\frac{2}{x y^{3} z^{2}}$.

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$$
\begin{aligned}
\begin{array}{ll}
\text { (a) } \mathrm{LCD}=2 x y^{5} z^{5} & \text { (b) } \mathrm{LCD}=2 y^{3} z^{3} \\
\text { (c) } \mathrm{LCD}=2 x y^{3} z^{3} & \text { (d) } \mathrm{LCD}=2 x y^{3} z^{5} \\
\text { 2. } \frac{x+y}{3 x^{3 / 2} y^{2}}-\frac{x^{2}+y^{2}}{6 x y^{4}} & \\
\begin{array}{ll}
\text { (a) } \mathrm{LCD}=18 x^{3 / 2} y^{4} & \text { (b) } \mathrm{LCD}=6 x^{3 / 2} y^{4} \\
\text { (c) } \mathrm{LCD}=18 x y^{4} & \text { (d) } \mathrm{LCD}=6 x y^{4}
\end{array}
\end{array}>.
\end{aligned}
$$

## End Quiz.

Now that you have seen a couple of examples in detail, you try. You have to be very simple minded! Apply the LCD Algorithm and use good algebraic techniques. Don't skip steps; do it once, right the first time!

ExErcise 4.12. Solve each of the following. I recommend you use the LCD Algorithm and the procedure outlined in Example 4.2 to obtain a least common denominator before combining fractions. Radicals can be converted to exponential notation. Negative exponents should be eliminated first.

Section 4: Basic Algebra, Part II

$$
\begin{array}{ll}
\text { (a) } \frac{4}{3 x^{2} y^{3}}-\frac{5}{4 x^{3} y} & \text { (b) } \frac{3 a}{4 b^{4}}+\frac{a+b}{9 a b} \\
\text { (c) } \frac{x^{2} y}{2(x+y)}-\frac{y}{4 x^{2}(x+y)} & \text { (d) } \frac{1}{x y^{5}}-\frac{2}{x^{3} y^{4}}+\frac{3}{x^{5} y^{3}}-\frac{4}{x^{7} y^{2}}
\end{array}
$$

## - Separating a Fraction

When we obtain a common denominator, we read and use equation (9) from left-to-right. The fundamental equation for adding fractions, equation (9), can also be read from right-to-left. This is a process of separating a fraction.

The equation

$$
\begin{equation*}
\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \tag{12}
\end{equation*}
$$

states that if we have a quotient and the numerator consists of the sum (or difference) of several terms, then the fraction can be broken up or separated according to (12).

This technique can be used to simplify an algebraic expression; more importantly, within the context of certain (Calculus) problems, using
this separation of fractions technique can advance your work toward a successful conclusion.

Illustration 10. Separate each of the following and simplify.

$$
\text { (a) } \frac{4 x^{2}+3}{2 x^{2}}=\frac{4 x^{2}}{2 x^{2}}+\frac{3}{2 x^{2}}=2+\frac{3}{2 x^{2}}=2+\frac{3}{2 x^{2}} \text {. }
$$

(b) Here's a common situation in which separation is useful.

$$
\begin{aligned}
\frac{4 x^{4}-3 x^{3}+2 x^{2}-x+1}{x} & =\frac{4 x^{4}}{x}-\frac{3 x^{3}}{x}+\frac{2 x^{2}}{x}-\frac{x}{x}+\frac{1}{x} \\
& =4 x^{3}-3 x^{2}+2 x-1+\frac{1}{x}
\end{aligned}
$$

(c) $\frac{4 y-1}{y+1}=\frac{4 y}{y+1}-\frac{1}{y+1}$.

Here are a couple of problems for you to consider.
Exercise 4.13. Separate each of the following and simplify.

Section 4: Basic Algebra, Part II

$$
\text { (a) } \frac{12 x^{3}-6 x^{2}+3}{3 x^{2}} \text { (b) } \frac{x+1}{x^{2}+1} \text { (c) } \frac{2(x+1)^{2}+6 x}{(x+1)}
$$

Common Error. An error students make in the heat of battle (a test) is the following one:

$$
\text { NOT TRUE! } \Longrightarrow \quad \frac{a}{b+c}=\frac{a}{b}+\frac{a}{c} \Longleftarrow \text { NOT TRUE! }
$$

This is called an algebraic blunder! Don't do this!

We've now finished Lesson 4. If you've stuck with me, congratulations! Follow me now to Lesson 5 to review multiplication and division of algebraic expressions.

## Solutions to Exercises

4.1. Solutions: Throughout we use the Law of Exponents without comment.
(a) $a b(c+d)=a b c+a b d$.
(b) $w^{-1}\left(4 w-3 w^{2}\right)=4-3 w$.
(c) $x^{3}\left(\sqrt{x}+x^{3 / 2}\right)=x^{3}\left(x^{1 / 2}+x^{3 / 2}\right)=x^{7 / 2}+x^{9 / 2}$.
(d) $\sqrt{w}(\sqrt{w}+3)=w+3 \sqrt{w}$.
(e) $s^{3}\left(5 s^{-4}-\frac{3}{s^{2}}\right)=5 s^{-1}-\frac{3 s^{3}}{s^{2}}=\frac{5}{s}-3 s$.
(f) Here is a bit more detail for this one

$$
\begin{aligned}
(s t)^{2}\left(s^{-1}+t^{-1}\right) & =s^{2} t^{2}\left(s^{-1}+t^{-1}\right) & & \triangleleft \text { Law Exp. \#2 } \\
& =s^{2} t^{2} s^{-1}+s^{2} t^{2} t^{-1} & & \triangleleft \text { Distrib. Law. } \\
& =s^{2-1} t^{2}+s^{2} t^{2-1} & & \triangleleft \text { Law Exp. \#1 } \\
& =s t^{2}+s^{2} t & & \triangleleft \text { arithmetic }
\end{aligned}
$$

A similar documentation can be done for problems (a)-(e); did you document?

Your answers may not jive exactly with what I have ... of course, you should continue to manipulate your answers to get mine.

Solutions to Exercises (continued)
4.2. Solution.

$$
\begin{aligned}
& s^{-1 / 2} t^{1 / 2}\left[\frac{1}{(s t)^{1 / 2}}+(s t)^{1 / 2}\right] \\
& \quad=\frac{s^{-1 / 2} t^{1 / 2}}{(s t)^{1 / 2}}+s^{-1 / 2} t^{1 / 2}(s t)^{1 / 2} \\
& \quad \triangleleft \text { Distrib. Law } \\
& \quad=\frac{s^{-1 / 2} t^{1 / 2}}{s^{1 / 2} t^{1 / 2}}+s^{-1 / 2} t^{1 / 2} s^{1 / 2} t^{1 / 2} \\
& \quad=\frac{1}{s^{1 / 2+1 / 2}}+s^{-1 / 2+1 / 2} t^{1 / 2+1 / 2} \\
& \quad=\frac{1}{s}+t
\end{aligned}
$$

We have used the fact that $s^{0}=1$ and $t^{0}=1$.

Solutions to Exercises (continued)
4.3. Solution:
(a) $4 w-12 w=-8 w$.
(b) $23 p^{2} q^{3}-6 p^{2} q^{3}=17 p^{2} q^{3}$.
(c) $\frac{1}{2} t^{3}+\frac{3}{2} t^{3}=2 t^{3}$.
(d) $\frac{3}{s^{2}}-\frac{6}{s^{2}}=-\frac{3}{s^{2}}$.
(e) $\frac{4 w}{\sqrt{s}}-\frac{3 w}{\sqrt{s}}=4 \frac{w}{\sqrt{s}}-3 \frac{w}{\sqrt{s}}=(4-3) \frac{w}{\sqrt{s}}=\frac{w}{\sqrt{s}}$.
(f) $4 x y-5 s t$ does not combine any further. : -)

Solutions to Exercises (continued)
4.4. Solutions:
(a) $3 w^{3} v^{4}+9 w^{2} v^{2}=3 w^{2} v^{2}\left(w v^{2}+3\right)$.
(b) $x^{2} \sqrt{y}+x^{3} y^{3 / 2}=x^{2} y^{1 / 2}+x^{3} y^{3 / 2}=x^{2} y^{1 / 2}(1+x y)$.
(c) $a b^{2} c^{3}+a^{3} b^{2} c=a b^{2} c\left(c^{2}+a^{2}\right)=a b^{2} c\left(a^{2}+c^{2}\right)$.
(d) $\frac{x y}{5}-\frac{x y}{10}=\frac{x y}{5}\left[1-\frac{1}{2}\right]=\frac{x y}{5} \frac{1}{2}=\frac{x y}{10}$.
(e) $(x+1) \sqrt{x}-x \sqrt{x}=[(x+1)-x] \sqrt{x}=\sqrt{x}$. Here have factored $\sqrt{x}$ out to the right. I hope that didn't psychologically disturb you.
(f) $\frac{2 p^{8} q^{4}}{w^{2}}+\frac{4 p^{8}}{w}=\frac{2 p^{8}}{w}\left[\frac{q^{4}}{w}+2\right]$.

Problem (f) can be simplified further, if you allow me to get a common denominator.

$$
\begin{aligned}
\frac{2 p^{8} q^{4}}{w^{2}}+\frac{4 p^{8}}{w} & =\frac{2 p^{8}}{w}\left[\frac{q^{4}}{w}+2\right] \\
& =\frac{2 p^{8}}{w} \frac{q^{4}+2 w}{w}=\frac{2 p^{8}\left(q^{4}+2 w\right)}{w^{2}}
\end{aligned}
$$

Solutions to Exercises (continued)
4.5. Solutions:
(a) $x^{8} y^{-2}-x^{-3} y^{-4}=x^{-3} y^{-4}\left(x^{11} y^{2}-1\right)$.
(b) $6 a^{3 / 2}-9 a^{-1 / 2}=3 a^{-1 / 2}\left(2 a^{2}-3\right)$.
(c) $12 w^{-3 / 2} s^{-12 / 5}+8 w^{-1 / 2} s^{-2 / 5}=4 w^{-3 / 2} s^{-12 / 5}\left(3+2 w s^{2}\right)$.

Comments: If you did not get these problems correct, be sure you understand why! The calculation of the exponents is always the problem.

Solutions to Exercises (continued)
4.6. Solutions:
(a) $\frac{a+a b}{a}=\frac{a(1+b)}{a}=1+b$.
(b) $\frac{x^{2} y^{4} z^{9}}{x^{3} y^{5} z^{3}}=\frac{z^{6}}{x y}$.
(c) $\frac{x^{-3} y^{-2}}{x^{4} y^{-5}}=\frac{y^{-2} y^{5}}{x^{4} x^{3}}=\frac{y^{3}}{x^{7}}$.
(d) $\frac{3 x^{3}+9 x^{2}}{3 x^{3}}=\frac{3 x^{2}(x+3)}{3 x^{3}}=\frac{x+3}{x}$.
(e) $\frac{4 x^{5 / 2}-x^{7 / 2}}{x^{2}}=\frac{x^{5 / 2}(4-x)}{x^{2}}=x^{1 / 2}(4-x)=\sqrt{x}(4-x)$.
(f) $\frac{y^{3} \sqrt{x}-y^{2} \sqrt{x}}{x y^{2} \sqrt{x}}=\frac{y^{2} \sqrt{x}(y-1)}{x y^{2} \sqrt{x}}=\frac{y-1}{x}$.

Cancellation in many cases is carried out by the subtraction of exponents. For example, in problem (e), we simplified the fraction

$$
\frac{x^{5 / 2}}{x^{2}}=x^{5 / 2} x^{-2}=x^{5 / 2-2}=x^{1 / 2}=\sqrt{x}
$$

Solutions to Exercises (continued)

If you become "expert" at these kinds of simplifications, you can jump for the left-most expression to the right-most expression in one "magnificent step."

Solutions to Exercises (continued)
4.7. Solutions:
(a) Simplify $\frac{x^{-3} y^{-2}-x^{2} y^{-5}}{x^{4} y^{-5}}$.

$$
\begin{aligned}
\frac{x^{-3} y^{-2}-x^{2} y^{-5}}{x^{4} y^{-5}} & =\frac{x^{-3} y^{-5}\left(y^{3}-x^{5}\right)}{x^{4} y^{-5}} & & \triangleleft \text { Tip } 1 \& \operatorname{Tip} 2 \\
& =\frac{x^{-3}\left(y^{3}-x^{5}\right)}{x^{4}} & & \triangleleft \text { cancel } \\
& =\frac{y^{3}-x^{5}}{x^{7}} . & & \triangleleft \text { Law of Exp. \#1 }
\end{aligned}
$$

The details of the other two problems are similar. I'll just present the answers. Mimic the solution to (a) to obtain correct solutions to (b) and (c) as necessary.
(b) $\frac{4 x^{-5 / 2}-x^{7 / 2}}{x^{1 / 2}}=\frac{4-x^{6}}{x^{3}}$
(c) $\frac{x^{2}}{4 x^{3}+2 x^{-3}}=\frac{x^{5}}{2\left(2 x^{6}+1\right)}$.

Solutions to Exercises (continued)
4.8. Solutions:
(a) $\frac{4 x^{2}}{5 y^{3}} \frac{x^{4} y^{2}}{x^{2} y}=\frac{4 x^{2} x^{4} y^{2}}{5 y^{3} x^{2} y}=\frac{4 x^{6} y^{2}}{5 x^{2} y^{4}}=\frac{4 x^{4}}{5 y^{2}}$.
(b) $(3 x-2)^{2} \frac{4 x}{(x+1)^{3}}=\frac{4 x(3 x-2)^{2}}{(x+1)^{3}}$
(c) $\frac{x}{y}\left[\frac{y}{z}+\frac{1}{2}\right]=\frac{x}{y} \frac{y}{z}+\frac{x}{y} \frac{1}{2}=\frac{x y}{y z}+\frac{x}{2 y}=\frac{x}{z}+\frac{x}{2 y}$.
(d) $\left[\frac{x^{4}}{(x+3)} \cdot \frac{x}{2}\right]^{-1}=\left[\frac{x^{5}}{2(x+3)}\right]^{-1}=\frac{2(x+3)}{x^{5}}$.
(e) $\left[x\left(\frac{x}{y}\right)^{-1}\right]^{-3}=\left[x\left(\frac{y}{x}\right)\right]^{-3}=\left[\frac{x y}{x}\right]^{-3}=y^{-3}=\frac{1}{y^{3}}$.

You should fully understand the justification of each step. Knowledge of the validity of each step will give you confidence in your algebraic abilities.

Solutions to Exercises (continued)
4.9. Solutions: We simply apply (8).
(a) $\frac{x /(x+y)}{x^{2} /(x+y)^{2}}=\frac{x(x+y)^{2}}{x^{2}(x+y)}=\frac{x+y}{x}$.
(b) $\frac{x^{3} y^{2} / s^{9} t^{2}}{x^{4} y^{8} / s^{10}}=\frac{x^{3} y^{2} s^{10}}{s^{9} t^{2} x^{4} y^{8}}=\frac{s}{x y^{6} t^{2}}$.
(c) $\frac{x^{4}}{x^{3} / 4}=\frac{4 x^{4}}{x^{3}}=4 x$.
(d) $\frac{x y /(x+y)}{(x+y)}=\frac{x y}{(x+y)^{2}}$.

In the case of (c), we have (mentally) written $\frac{x^{4}}{x^{3} / 4}$ as $\frac{x^{4} / 1}{x^{3} / 4}$ and then applied (8).
Ditto for part (d). Write $\frac{x y /(x+y)}{(x+y)}$ as $\frac{x y /(x+y)}{(x+y) / 1}$ and apply (8).

Solutions to Exercises (continued)
4.10. Solutions:
(a) $\frac{x}{\sqrt{7}}=\frac{\sqrt{7} x}{7}$.
(b) $\frac{3 x^{2}}{\sqrt{x+1}}=\frac{3 x^{2} \sqrt{x+1}}{x+1}$.
(c) $\frac{x y}{\sqrt[4]{3}}=\frac{x y}{3^{1 / 4}}=\frac{3^{3 / 4} x y}{3}=\frac{\left(3^{3}\right)^{1 / 4} x y}{3}=\frac{\sqrt[4]{27} x y}{3}$.

Solutions to Exercises (continued)
4.11. Solutions:
(a) $\frac{2}{a}+\frac{a}{2}=\frac{2}{a} \frac{2}{2}+\frac{a}{2} \frac{a}{a}=\frac{4}{2 a}+\frac{a^{2}}{2 a}=\frac{4+a^{2}}{2 a}$
(b) We proceed as follow:

$$
\begin{aligned}
\frac{3 x}{2 x-y}+\frac{y}{x} & =\frac{3 x}{2 x-y} \frac{x}{x}+\frac{y}{x} \frac{2 x-y}{2 x-y}=\frac{3 x^{2}}{x(2 x-y)}+\frac{y(2 x-y)}{x(2 x-y)} \\
& =\frac{3 x^{2}+y(2 x-y)}{x(2 x-y)}=\frac{3 x^{2}+2 x y-y^{2}}{x(2 x-y)}
\end{aligned}
$$

(c) Let's use standard techniques.

$$
\begin{aligned}
\frac{1}{2 y}-\frac{x}{3}+\frac{y}{(1+x)} & =\frac{1}{2 y} \frac{3(x+1)}{3(x+1)}-\frac{x}{3} \frac{2 y(x+1)}{2 y(x+1)}+\frac{y}{(1+x)} \frac{6 y}{6 y} \\
& =\frac{3(x+1)-2 x y(x+1)+6 y^{2}}{6 y(x+1)} \\
& =\frac{-2 x^{2} y-2 x y+6 y^{2}+3 x+3}{6 y(x+1)}
\end{aligned}
$$

Solutions to Exercises (continued)
4.12. Solutions: Fill in the details please!
(a) $\frac{4}{3 x^{2} y^{3}}-\frac{5}{4 x^{3} y}=\frac{16 x-15 y^{2}}{12 x^{3} y^{3}}$
(b) $\frac{3 a}{4 b^{4}}+\frac{a+b}{9 a b}=\frac{(3 a)(9 a)+(a+b)\left(4 b^{3}\right)}{36 a b^{4}}=\frac{27 a^{2}+4 a b^{3}+4 b^{4}}{36 a b^{4}}$
(c) Here the LCD $=4 x^{2}(x+y)$. Thus,

$$
\begin{aligned}
\frac{x^{2} y}{2(x+y)}-\frac{y}{4 x^{2}(x+y)} & =\frac{\left(x^{2} y\right)\left(2 x^{2}\right)-y}{4 x^{2}(x+y)} \\
& =\frac{2 x^{4} y-y}{4 x^{2}(x+y)}
\end{aligned}
$$

(d) The LCD $=x^{7} y^{5}$. Thus,

$$
\frac{1}{x y^{5}}-\frac{2}{x^{3} y^{4}}+\frac{3}{x^{5} y^{3}}-\frac{4}{x^{7} y^{2}}=\frac{x^{6}-2 x^{4} y+3 x^{2} y^{2}-4 y^{3}}{x^{7} y^{5}}
$$

Solutions to Exercises (continued)
4.13. Solutions:
(a) $\frac{12 x^{3}-6 x^{2}+3}{3 x^{2}}=\frac{12 x^{3}}{3 x^{2}}-\frac{6 x^{2}}{3 x^{2}}+\frac{3}{3 x^{2}}=4 x-2+\frac{1}{x^{2}}$
(b) $\frac{x+1}{x^{2}+1}=\frac{x}{x^{2}+1}+\frac{1}{x^{2}+1}$
(c) $\frac{2(x+1)^{2}+6 x}{(x+1)}=\frac{2(x+1)^{2}}{(x+1)}+\frac{6 x}{(x+1)}=2(x+1)+\frac{6 x}{(x+1)}$

## Solutions to Examples

4.1. Solutions:
(a) Combine $6 x-\frac{3}{x}$.

$$
\begin{aligned}
6 x-\frac{3}{x} & =(6 x) \frac{x}{x}-\frac{3}{x}=\frac{6 x^{2}}{x}-\frac{3}{x} & & \triangleleft \text { get common denom. } \\
& =\frac{6 x^{2}-3}{x}=\frac{3\left(2 x^{2}-1\right)}{x} & & \triangleleft \text { combine }
\end{aligned}
$$

(b) Combine $\frac{1}{\sqrt{x}}-\sqrt{x}$.

$$
\begin{aligned}
\frac{1}{\sqrt{x}}-\sqrt{x} & =\frac{1}{\sqrt{x}}-\sqrt{x} \frac{\sqrt{x}}{\sqrt{x}} & & \triangleleft \text { get comm } \\
& =\frac{1}{\sqrt{x}}-\frac{x}{\sqrt{x}}=\frac{1-x}{\sqrt{x}} & & \triangleleft \text { combine }
\end{aligned}
$$

Solutions to Examples (continued)
(c) Combine $\frac{x^{2}}{x+y}-\frac{x}{y}$.

$$
\begin{aligned}
\frac{x^{2}}{x+y}-\frac{x}{y} & =\frac{x^{2}}{x+y} \frac{y}{y}-\frac{x}{y} \frac{(x+y)}{(x+y)} & \triangleleft \text { get common denom. } \\
& =\frac{x^{2} y}{(x+y) y}-\frac{x(x+y)}{y(x+y)} & \triangleleft \text { common denom. attained! } \\
& =\frac{x^{2} y-x(x+y)}{(x+y) y} & \triangleleft \text { combine } \\
& =\frac{x^{2} y-x^{2}-x y}{(x+y) y} &
\end{aligned}
$$

(d) Combine $\frac{2}{x}-\frac{3 y}{a}+\frac{z}{2 b}$.

$$
\begin{aligned}
\frac{2}{x}-\frac{3 y}{a}+\frac{z}{2 b} & =\frac{2}{x} \frac{2 a b}{2 a b}-\frac{3 y}{a} \frac{2 x b}{2 x b}+\frac{z}{2 b} \frac{a x}{a x} \quad \triangleleft \text { get common denom. } \\
& =\frac{4 a b-6 b x y+a x z}{2 a b x}
\end{aligned}
$$

Solutions to Examples (continued)

Example Notes: The common denominator is obtained by multiplying all the denominators together. Once the common denominator has been identified, we multiply the numerator and denominator of each term by 1 in a form that will produce the common denominator. (This should be clear!)

- Examples (a)-(c) demonstrate the procedure for combining two fractions. In example (d), the same procedure is used for obtaining a common denominator for the sum of three terms.
- This is the same procedure that you have been using for years to combine numerical fractions: $1 / 3-1 / 2=2 / 6-3 / 6=-1 / 6$.
- This is a very important and basic skill. Combining is simplicity itself as long as you understand the underlying reasoning-that of multiplying each term by an appropriately chosen version of 1 . Whenever you get confused and/or disoriented about what to do, the problem reduced down to multiplying each term by 1!
- You can always obtain a common denominator by multiplying all the denominators together. Usually though, we try to obtain the least common denominator! Details to follow.

Solutions to Examples (continued)
4.2. To find the LCD of

$$
\frac{1}{4 x}-\frac{4}{x^{3} y}+\frac{3 x}{2 y^{4}}
$$

I'll follow my own advice.
Step 1: Factor each denominator completely. In this example, this step is easy, though Step 1 has the potential of being the most difficult step.

First term has denominator: $4 x \quad$ Factors: $2^{2}, x$
Second term has denominator: $x^{3} y$ Factors: $x^{3}, y$
Third term has denominator: $2 y^{4} \quad$ Factors: $2, y^{4}$
Step 2: Write down all factors. The factors are

$$
2^{2}, x, x^{3}, y, 2, y^{4}
$$

Step 3: Remove lowest powers. Let's rearrange the previous display:

$$
\underbrace{2,2^{2}}, \underbrace{x, x^{3}}, \underbrace{y, y^{4}}
$$

same base same base same base

Within each group, we eliminate all but the one with the highest exponent to obtain

$$
2^{2}, x^{3}, y^{4}
$$

Step 4: Multiply all the factors together left after step 3 to obtain the least common denominator. According to my LCD "algorithm," the least common denominator is

$$
\mathrm{LCD}=2^{2} x^{3} y^{4}=4 x^{3} y^{4}
$$

Now that we have jumped through hoops to get the LCD, we still have the job of combining the expression.

Solutions to Examples (continued)
Now combine!

$$
\begin{aligned}
\frac{1}{4 x}-\frac{4}{x^{3} y}+\frac{3 x}{2 y^{4}} & =\frac{1}{4 x}\left[\frac{x^{2} y^{4}}{x^{2} y^{4}}\right]-\frac{4}{x^{3} y}\left[\frac{4 y^{3}}{4 y^{3}}\right]+\frac{3 x}{2 y^{4}}\left[\frac{2 x^{3}}{2 x^{3}}\right] \\
& =\frac{x^{2} y^{4}}{4 x^{3} y^{4}}-\frac{16 y^{3}}{4 x^{3} y^{4}}+\frac{6 x^{4}}{4 x^{3} y^{4}} \\
& =\frac{x^{2} y^{4}-16 y^{3}+6 x^{4}}{4 x^{3} y^{4}}
\end{aligned}
$$

Presentation of Answer:

$$
\frac{1}{4 x}-\frac{4}{x^{3} y}+\frac{3 x}{2 y^{4}}=\frac{x^{2} y^{4}-16 y^{3}+6 x^{4}}{4 x^{3} y^{4}}
$$

That was easy!
Once you understand and understand the LCD Algorithm, the process of getting the least common denominator will be ... second nature.

Example 4.2.
4.3. Let's delineate the details one more time! Our given expression is

$$
\frac{2 \sqrt{x}}{5 y^{2}}+\frac{3}{2 x^{1 / 2} y^{3 / 2}}
$$

Step 1: Factor each denominator completely.
First term has denominator: $5 y^{2} \quad$ Factors: 5, $y^{2}$
Second term has denominator: $2 x^{1 / 2} y^{3 / 2}$ Factors: $2, x^{1 / 2}, y^{3 / 2}$
Step 2: Write down all factors. The factors are

$$
5, y^{2}, 2, x^{1 / 2}, y^{3 / 2}
$$

Step 3: Remove lowest powers. Let's rearrange the previous display:

$$
2,5, x^{1 / 2}, y^{2}, y^{3 / 2}
$$

Within each group of factors having the same base, we eliminate all but the one with the highest exponent to obtain

$$
2,5, x^{1 / 2} y^{2}
$$

Step 4: Multiply all the factors together left after step 3 to obtain the least common denominator. According to my LCD "algorithm," the least common denominator is

$$
\mathrm{LCD}=(2)(5) x^{1 / 2} y^{2}=10 x^{1 / 2} y^{2}
$$

Now combine!

$$
\begin{aligned}
\frac{2 \sqrt{x}}{5 y^{2}}+\frac{3}{2 x^{1 / 2} y^{3 / 2}} & =\frac{2 x^{1 / 2}}{5 y^{2}}\left[\frac{2 x^{1 / 2}}{2 x^{1 / 2}}\right]+\frac{3}{2 x^{1 / 2} y^{3 / 2}}\left[\frac{5 y^{1 / 2}}{5 y^{1 / 2}}\right] \\
& =\frac{4 x}{10 x^{1 / 2} y^{2}}+\frac{15 y^{1 / 2}}{10 x^{1 / 2} y^{2}}
\end{aligned}
$$

Presentation of Answer:

$$
\frac{2 \sqrt{x}}{5 y^{2}}+\frac{3}{2 x^{1 / 2} y^{3 / 2}}=\frac{4 x+15 y^{1 / 2}}{10 x^{1 / 2} y^{2}}
$$

And that's the procedure! Do you think you can do it?

## Important Points

If you erred on this one, more than likely it was on the appropriate multiplicative constant: 6 not 18. At least that's what I'm betting on.

The instructions of the LCD Algorithm said to completely factor the denominator. Here's a list of the factors


Let's rearrange them

$$
2,3,3, x, x^{3 / 2}, y^{2}, y^{4}
$$

Now drop duplicate factors-that's the 3. Oops! I did mention dropping identical factors, didn't I?

$$
2,3, x, x^{3 / 2}, y^{2}, y^{4}
$$

Now, from each group all members of which have the same base, drop all but the one with the highest power.

$$
2,3, x^{3 / 2}, y^{4}
$$

Important Points (continued)
and the LCD is the product of same:

$$
\mathrm{LCD}=(2)(3) x^{3 / 2} y^{4}=6 x^{3 / 2} y^{4}
$$

Alternative (a) will work as a common denominator, but it is not the least common denominator. If you use (a), you will be working with larger numbers than is really necessary.


[^0]:    ${ }^{1}$ Pardon my French. Translation: Multiplying fractions together is a (piece) of cake.

